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EQUIVALENCE AMONG OPTIMIZATION PROBLEMS ON MATRIX SETS

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Treatment of optimization problems on matrix sets is a general framework for the study of some large classes of discrete programming problems, for the investigation of connections between different classes of such problems. An appropriate formalism is introduced. It gives a possibility to include in this study bottle-neck problems and other combinatorial optimization problems over totally ordered commutative semigroups. Concepts of equivalency and of weak equivalency are defined and some general equivalency theorems are proved. The main problem under discussion is for which problems an equivalent problem over a finite ordered algebraic structure can be constructed.

1. Introduction

Many well-known discrete programming problems (such as optimization problems on subsets of a set or on subgraphs of a graph, optimization problems on permutation sets etc.) can be treated as optimization problems on matrix sets. Such an approach does not yield so much for any single problem, but it is important as a general framework for the study of some large classes of discrete programming problems and for the investigation of connections between such classes. To include in this study most of the interesting cases of optimization such as, for example, bottle-neck problems, we shall use the following formalism.

Let A be a totally ordered commutative semigroup (toc-semigroup) with the operation $+$ and $A'=A\cup\{\Lambda\}$, where $a+\Lambda=\Lambda+a=a$, $\Lambda\le a$ for any $a\in A$. Then for any $m\times n$-matrix $C=\|c_{ij}\|$, $c_{ij}\in A$, and for any zero-one $m\times n$-matrix $U=\|u_{ij}\|$, $u_{ij}\in\{0,1\}$, we can define the following scalar product of matrices:

$$CU=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}u_{ij},$$

where $c_{ij}1=c_{ij}$, $c_{ij}0=\Lambda$. We therefore have that $CU\in A'$ and even $CU\in A$ if $U\neq 0$.

Let $\mathscr{A}$ be a set of zero-one $m\times n$-matrices and let the matrix C be fixed. We shall consider the following general discrete optimization problem:

$$\text{find } U_0\in\mathscr{A} \quad \text{for which} \quad CU_0=\max_{U\in\mathscr{A}} CU. \tag{1.1}$$

This problem will be called the *$(\mathscr{A}, A, C)$-problem* (or simply the *$\mathscr{A}$-problem* if A and

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C are known or not essential) and the matrix C will be called the *weight matrix* of this problem. So we can say that CU is the *C-weight* of the matrix U. Every element $U_0 \in \mathscr{A}$ with minimum C-weight (1.1) will be called a *C-optimal element* of the $(\mathscr{A}, A, C)$-problem.

Traditionally, several special $(\mathscr{A}, A, C)$ problems were considered in the case that A is the additive semigroup R of all real numbers. But an interesting case such as the bottleneck problem (for which A must be a commutative semigroup with the addition $a+b=\min(a,b)$ defined on a totally ordered set) demonstrates that by considering arbitrary toc-semigroups we can immensely expand the class of practical optimization problems included in the general scheme. Another example: any problem of finding the element $U_* \in \mathscr{A}$ with maximum C-weight is equivalent (in a natural sense) to the $(\mathscr{A}, A, C)$-problem, where A^* is the semigroup A with the dual order. Such a case occurs for example when we want to find a path with maximum channel capacity in a network.

We note that the general scheme introduced above makes it possible to classify discrete programming problems by their parameters $\mathscr{A}$, A and C. Examples of problems with different sets $\mathscr{A}$ will be given in the next section. Some examples of problems with different semigroups A have been discussed above. About the parameter C we shall say that by narrowing the class of weight-matrices essentially we can extract the so-called well-solvable cases of discrete optimization problems studied by many authors.

2. Examples

We shall choose from the big variety of discrete programming problems on matrix sets some essential groups of problems. Here shall be $m=1$ or $m=n$. As an example with $m \neq n$ let us name the $m \times n$-assignment problem.

2.1. *Optimization on subsets*

For a given set $S=\{s_1, \dots, s_n\}$ a collection T of its subsets is selected. Each $s_i \in S$ has a weight $c_i \in A$. Then for an element $W \in T$ the weight of W is equal to the sum of the weights of all elements $s_i \in W$. The aim is to find a set $W \in T$ with the minimum weight.

This problem can be treated as an $(\mathscr{A}, A, C)$-problem for which $m=1$, $C=\|c_1, \dots c_n\|$ and $\mathscr{A}$ consists of all characteristic vectors of the sets $W \in T$.

If T is the collection of all one-element subsets of S, we obtain the general mathematical programming problem on an arbitrary set. In general, the case with $m=1$ can be treated as a vector optimization problem. As examples we consider here the integer linear programming model

$$\sum_{i=1}^{n} c_i x_i \to \min,$$

$$\sum_{i=1}^{n} a_{ij} x_i \leq b_j \quad (j=1,\ldots,l),$$

$$a_{ij} \geq 0,\ b_j \geq 0,\ c_i \geq 0 \quad (i=1,\ldots,n; j=1,\ldots,l),$$

$$x_i \text{ integer} \quad (i=1,\ldots,n),$$

or even the integer programming model with (partly) nonlinear constraints but with a linear objective function.

2.2. *Optimization on subgraphs*

For a weighted digraph G a collection T of subgraphs is considered. An arc-weight matrix $D=\|d_{ij}\|$ is given, where d_{ij} is the weight of the arc (i,j). The weight of a subgraph is defined as the sum of weights of all its arcs. The aim is to find a subgraph from T with minimum weight.

This problem can be treated as an $(\mathscr{A}, A, C)$-problem for which $m=n$; $C=D$ and $\mathscr{A}=\mathscr{A}_T$ is the set of vertex-to-vertex *adjacency* matrices of all subgraphs from T.

In this class of $(\mathscr{A}, A, C)$-problems, the following optimization problems are best known:

(1) the minimum spanning tree problem;
(2) the shortest path problem;
(3) the open circuit travelling salesman problem;
(4) the Chinese postman problem.

2.3. *Optimization on permutation sets*

Taking a set $J=\{1,\ldots,n\}$ and the set S_n of all its permutations $s: i \to s(i)$ with $i, s(i) \in J$, a subset $H \subset S_n$ is selected and a distance matrix $D=\|d_{ij}\|$ is given. The D-length $L_D(s)$ of an element $s \in S_n$ is defined by $L_D(s)=\sum_{i=1}^{n} d_{is(i)}$. The aim is to find an $h_0 \in H$ for which $L_D(h_0)=\min_{s \in H} L_D(s)$. This general problem can be treated as an $(\mathscr{A}, A, C)$-problem for which $m=n$, $C=D$, $\mathscr{A}$ is the set of permutation matrices of all $s \in H$ and the C-weight is equal to the D-length. The best known examples of optimization problems on permutation sets are the following:

(1) the linear assignment problem;
(2) the travelling salesman problem;
(3) the symmetric assignment problem;
(4) various problems of many salesmen (for example, the problem of three salesmen, the first of which must visit two towns, the second three and the third all other towns).

3. Ordering of optimization problems

For two quasiorders α and β on a set S we shall write $\alpha \sigma \beta$ if and only if $s \alpha t \Rightarrow s \beta t$ for any two elements $s, t \in S$, i.e. if and only if the relation α is a subset of the relation β. In this way we obtain a (partially) ordered set of all quasiorders on a given set. Maximal elements of such an ordered set are total quasiorders, i.e. quasiorders γ such that for any two elements a and b we have $a\gamma b$ or $b\gamma a$. If we restrict ourselves to the class of orders then maximal elements are total orders.

Taking an $(\mathscr{A}, A, C)$-problem $\hat{\alpha}$ we obtain on the set $\mathscr{A}$ of binary matrices a quasiorder α which can be defined in the following way: for two matrices $U, V \in \mathscr{A}$ we have $U\alpha V$ if and only if $CU \leq CV$. The problem $\hat{\alpha}$ is called weaker than problem $\hat{\beta}$ (and we write $\hat{\alpha}\hat{\sigma}\hat{\beta}$) if $\alpha\sigma\beta$ in the above sense. In this case the problem $\hat{\beta}$ is called stronger than problem $\hat{\alpha}$.

If an $(\mathscr{A}, A, C)$-problem is weaker than an $(\mathscr{A}, A, D)$-problem then any D-optimal element is C-optimal.

This statement can be useful in solving practical optimization problems by changing the problem into a stronger one that will be easier to solve.

The relation $\hat{\sigma}$ is a quasiorder on the set of all $\mathscr{A}$-problems. It is not always an order, because for example $\alpha\sigma\beta$ and $\beta\sigma\alpha$ for the $(\mathscr{A}, R, C)$-problem $\hat{\alpha}$ and the $(\mathscr{A}, R, D)$-problem $\hat{\beta}$ with $D = \|c_{ij} + a\|$ for some $a \in A$, while $\hat{\alpha} \neq \hat{\beta}$ if $a \neq 0$. But the problems $\hat{\alpha}$ and $\hat{\beta}$ introduce the same quasi-order on the set $\mathscr{A}$: $\alpha = \beta$. We shall consider such problems as equivalent. Our main purpose is to prove equivalence theorems for $\mathscr{A}$-problems.

4. Equivalence theorem

We consider in this section only matrices $U = \|u_{ij}\|$, $i, j \in J$ with the property that

$$\sum_{i=1}^{m} \sum_{i=1}^{m} u_{ij} = k = \text{constant}. \tag{4.1}$$

For all examples from subsection 2.3 property (1) is valid and $k = n$ in this case.

Another example (from subsection 2.2): $\mathscr{A}$ is the set of subgraphs of a digraph, each of which has exactly k edges. And a third example (from subsection 2.1): a collection T of subsets of a set S in which any set $t \in T$ has exactly k elements.

An $(\mathscr{A}, A, C)$-problem is called *equivalent* to an $(\mathscr{A}, B, D)$-problem if and only if for any two matrices, $U, V \in \mathscr{A}$

$$CU < CV \Leftrightarrow DU < DV.$$

The following result generalized Theorem 1 of [7].

Theorem 1 (General Equivalence Theorem). *Any $(\mathscr{A}, A, C)$-problem with property* (4.1) *is equivalent to an $(\mathscr{A}, B, D)$-problem over a finite totally ordered commutative semigroup B.*

Proof. The case $k = 0$ is trivial. For $k \geq 1$ let us consider the lexicographically ordered Cartesian product (the lexicographic product)

$$A \times \mathbb{N} = \{(a, l) \colon a \in A, l \in \mathbb{N}\},$$

where $\mathbb{N}$ is the semigroup of all natural numbers ordered in the usual way. Let G be the subsemigroup of $A \times \mathbb{N}$ generated by the elements $(c_{ij}, 1)$ for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Let B denote the Rees factor-seimgroup G/F, where F is the convex ideal $\{(a, l) \colon a \in G, l > k\}$.

We define the $m \times n$-matrix $D = \|d_{ij}\|$, $d_{ij} \in B$ by $d_{ij} = (c_{ij}, 1)$. Then $DU = (CU, k)$, $DU \in B$. It is clear that the $(\mathscr{A}, A, C)$-problem is equivalent to the $(\mathscr{A}, B, D)$-problem and that the semigroup B is finite.

Corollary 1. *The equivalence theorem holds for the following problems:*

(1) *the minimum spanning tree problem* $(k = n - 1)$*;*

(2) *the linear assignment problem* $(k = n)$*;*

(3) *the travelling salesman problem* $(k = n)$*;*

(4) *the symmetric assignment problem* $(k = n)$*;*

(5) *the problem of* p $(1 \leq p \leq n)$ *salesmen with disjoint tours* $(k = n)$ *and a common initial point* $(k = p + n - 1)$*;*

(6) *the milti-salesmen problem with disjoint initial points, in which the number of cities to be visited by each salesman is given* $(k = n)$*;*

(7) *the open circuit travelling salesman problem* $(k = n - 1)$.

5. A counterexample

It is natural to ask if Theorem 1 can be extended to the general case or not. We shall demonstrate that it is impossible.

Let $\mathbb{Z}$ be the additive semigroup of all integers.

Theorem 2. *If property* (1) *does not hold, then for any finite totally ordered commutative semigroup B there exists an $(\mathscr{A}, \mathbb{Z}, C)$-problem not equivalent to any $(\mathscr{A}, B, D)$-problem.*

Proof. We shall prove this statement by constructing a counterexample. Let G be the digraph with adjacency matrix $\begin{Vmatrix} 1 & 1 \\ 1 & 0 \end{Vmatrix}$. Let us consider the subgraphs of G:

$$U_1 = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \qquad U_2 = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \qquad U_3 = \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix}, \qquad U_4 = \begin{Vmatrix} 1 & 1 \\ 0 & 0 \end{Vmatrix}.$$

We define the matrix C as follows:

$$C = \begin{Vmatrix} 1 & 0 \\ -1 & 0 \end{Vmatrix}.$$

Then for the elements of the set $\mathscr{A} = \{U_1, U_2, U_3, U_4\}$ we have

$$CU_1 = CU_4 = 1 > CU_2 = CU_3 = 0.$$

Now let us assume that there exist a finite ordered semigroup B and a matrix

$$D=\begin{Vmatrix} x & v \\ y & z \end{Vmatrix}, \quad x, y, z, v \in B,$$

such that the $(\mathscr{A}, \mathbb{Z}, C)$-problem is equivalent to the $(\mathscr{A}, B, D)$-problem.

Then from the definition of equivalence of two problems it follows that $x = DU_1 = DU_4 = x+z > z = DU_2 = DU_3 = x+y$. Since the samigroup B is finite, a minimum natural number k exists for which $kx = (k+1)x$. Hence if $k=1$ we have $x = 2x$ and $z = x+y = 2x+y = x+(x+y) = x+z$, but this contradicts the inequality $x+z>z$. If $k \geq 2$ we have $kx = (k-1)x+x = (k-1)x+(x+z) = kx+z = kx+(x+y) = (k+1)x+y = kx+y = (k-1)x+(x+y) = (k-1)x+z = (k-2)x+(x+z) = (k-2)x+x = (k-1)x$, but this contradicts the minimality of k. Therefore no finite B exists for which our $(\mathscr{A}, A, C)$-problem is equivalent to an $(\mathscr{A}, B, D)$-problem.

6. Weak equivalence

In this section we shall show that it is possible to weaken the concept of equivalence, preserving most of its useful qualities, so as to obtain a positive solution to the problem of extending Theorem 1 to the general case.

The $(\mathscr{A}, A, C)$- and $(\mathscr{B}, B, D)$-problems are called weakly equivalent if there exists a bijection $\tau : \mathscr{A} \to \mathscr{B}$ such that for any two matrices $U, V \in \mathscr{A}$

$$CU < CV \Leftrightarrow D(U\tau) < D(V\tau).$$

In fact, the concept of equivalence was first introduced by D.A. Suprenenko in [1] in exactly this way.

Theorem 3. *Each $(\mathscr{A}, A, C)$-problem is weakly equivalent to a $(\mathscr{B}, B, D)$-problem over a finite totally ordered commutative semigroup B.*

Proof. Let $\mathscr{C}$ be the set of all cell-matrices of the form

$$\| W_1 \vdots W_2 \|,$$

where W_1 and W_2 are zero–one $m \times n$-matrices. Let C' be the cell-matrix of the form

$$C' = \| C \vdots \Omega \|,$$

where Ω is the $m \times n$ matrix all elements of which are equal to Λ.

Now we define the set $\mathscr{B}$ as the following subset of the set $\mathscr{C}$:

$$\mathscr{C} = \{ U\tau \mid U \in \mathscr{A} \}$$

where

$$U\tau = \| U \vdots U^1 \|, \quad U^1 = \| u_{ij}^1 \|, \quad u_{ij}^1 = 1 - u_{ij} \quad (i = 1, \ldots, m; j = 1, \ldots, n).$$

Of course, τ is a bijection.

As $C'(U\tau)=CU$, it is clear that the original $(\mathscr{A}, A, C)$-problem is weakly equivalent to the constructed $(\mathscr{B}, A', C')$-problem.

For any matrix $V\in\mathscr{C}$, $V=\|v_{ij}\|$

$$\sum_{i=1}^{m}\sum_{j=1}^{2n} v_{ij}=nm.$$

Therefore we can apply Theorem 1 for $k=nm$. Hence the $(\mathscr{B}, A', C')$-problem is equivalent to $(\mathscr{B}, B, D)$-problem over a finite totally ordered commutative semigroup B. But then our $(\mathscr{A}, A, C)$-problem is also weakly equivalent to this $(\mathscr{C}, B, C)$-problem.

7. Reduction to integers in the case of real matrices

Let $\mathbb{R}$ and $\mathbb{Q}$ be the additive semigroups of all real and rational numbers respectively. We remember that $\mathbb{Z}$ and $\mathbb{N}$ are the additive semigroups of integer and natural numbers respectively. All these semigroups are assumed to be ordered in the usual way. In this section we assume that $A=\mathbb{R}$.

Theorem 4. *Each $(\mathscr{A}, \mathbb{R}, C)$-problem is equivalent to an $(\mathscr{A}, \mathbb{Z}, D)$-problem.*

Proof. The proof is divided into parts I–IV. In I we give some remarks about linear equation systems. In II and III an $\mathscr{A}$-problem over Q is constructed equivalent to the initial one. The lemma proved in II is used in III to demonstrate the equivalence. In IV some concluding remarks are made.

I. For an arbitrary $(\mathscr{A}, \mathbb{R}, C)$-problem let

$$U_1, \ldots, U_k \tag{7.1}$$

be the list of all matrices from $\mathscr{A}$. We assume $k\geq 1$. Let $CU_l=L_l$, $l=1,\ldots,k$ and $L_1\leq\cdots\leq L_k$. Then the system of linear equations

$$f_l(x_{11}, x_{12}, \ldots, x_{mn})=L_l \quad (l=1,\ldots k) \tag{7.2}$$

with coefficients 0 and 1, where

$$f_l=U_lX, \qquad x=\|x_{ij}\| \quad (i=1,\ldots,m,\ j=1,\ldots,n) \tag{7.3}$$

has the following solution:

$$x_{ij}=c_{ij} \quad (i=1,\ldots,m;\ j=1,\ldots,n) \tag{7.4}$$

in real numbers.

Some of the variables x_{ij} in the system (7.2) have only zero coefficients. These are fictitious variables. We are interested only in actual variables of the system (7.2). Therefore, it is useful to change our notation. We shall denote the variables of the functions (7.3) or, equivalently, of system (7.2) as $x_1, x_2, \ldots, x_w$ and rewrite system

(7.2) in the form

$$f_l(x_1, \ldots, x_w) = L_l \quad (l = 1, \ldots, k), \tag{7.5}$$

where the functions (7.3) have the form

$$f_l = a_{l1}x_1 + \cdots + a_{lw}x_w$$

and $a_{lu} \in \{0, 1\}$ $(l = 1, \ldots, k;\ u = 1, \ldots, w)$. The solution (7.4) can then be rewritten in the form

$$x_u = c_u \quad (u = 1, \ldots, w). \tag{7.6}$$

Now we transform system (7.5) as follows. Let

$$L_1 = \cdots = L_{k_1} \neq L_{k_1+1} = \cdots = L_{k_2} \neq L_{k_2+1} = \cdots$$

(in the case that $L_1 = \cdots = L_k$ the $(\mathscr{A}, \mathbb{R}, C)$-problem is equivalent to the H-problem over $\{0\}$ and hence over $\mathbb{Z}$). If $L_1 \neq 0$ we replace the equations $f_1 = L_1, \ldots, f_{k_1} = L_{k_1}$ of system (7.5) by

$$f_1 = L_1, \quad f_2 - f_1 = 0, \ \ldots, \ f_{k_1} - f_1 = 0, \tag{7.7}$$

then if $L_{k_1+1} \neq 0$ the equations $f_{k_1+1} = L_{k_1+1}, \ldots, f_{k_2} = L_{k_2}$ by

$$f_{k_1+1} = L_{k_1+1}, \quad f_{k_1+2} - f_{k_1+1} = 0, \ \ldots, \ f_{k_1+1} = 0, \quad \text{and so on} \tag{7.7}$$

It is clear that the new system (7.7) has the same solution (7.6).

Let us consider the maximal homogeneous subsystem of system (7.7). This homogeneous subsystem may be rewritten in the form

$$b_{l1}x_{j_1} - \cdots - b_{lp}x_{j_p} = 0 \quad (l = 1, \ldots, q), \tag{7.8}$$

where any variable

$$x_{j_1}, \ldots, x_{j_p} \tag{7.9}$$

has coefficient 1 in at least one of the equations (7.8).

II. Let Δ be the minimum among the numbers

$$L_{k_1+1} - L_{k_1}, L_{k_1+1} - L_{k_2}, \ldots$$

Lemma 1. *There exists a collection*

$$d_1, \ldots, d_w \tag{7.10}$$

of rational numbers, such that

$$d_u = c_u + \varepsilon_u, \qquad |\varepsilon_u| < \Delta/3mn \quad (u = 1, \ldots, w) \tag{7.11}$$

and $x_{j_s} = d_{j_s}$ $(s = 1, \ldots, p)$ *is a solution to system* (7.8).

Proof. If the rank r of system (7.8) is equal to p, then (7.8) has only the tivial solution $x_{j_s} = 0$ $(s = 1, \ldots, p)$. In this case we get $d_{j_s} = \varepsilon_{j_s} = 0$ for $s = 1, \ldots, p$ and assign to the other numbers (7.10) any arbitrary value satisfying condition (7.11).

If $r<p$ then $p-r$ variables may be considered as free variables. To be more concrete we shall assume that the first $p-r$ variables are considered as free ones. Then any other variable x_{j_s}, $p-r<s\le p$ may be represented as a linear form of free variables with integer coefficients:

$$x_{j_s} = a_1(s)x_{j_1} + \cdots + a_{p-r}(s)x_{j_{p-r}} \tag{7.12}$$

Let $M_s = |a_1(s)| + \cdots + |a_{p-r}(s)|$ and $M = \max_{p-r<s\le p} M_s$. Let d_{j_s} $(s=1,\dots,p-r)$ be arbitrary rational numbers for which

$$d_{j_s} = c_{j_s} + \varepsilon_{j_s}, \qquad |\varepsilon_{j_s}| < \Delta/3mnM \quad (s=1,\dots,p-r). \tag{7.13}$$

For $p-r<s\le p$ we define

$$\begin{aligned} d_{j_s} &= a_1(s)d_{j_1} + \cdots + a_{p-r}(s)d_{j_{p-r}} \\ &= a_1(s)c_{j_1} + \cdots + a_{p-r}(s)c_{j_{p-r}}) + (a_1(s)\varepsilon_{j_1} + \cdots + a_{p-r}(s)\varepsilon_{j_{p-r}}). \end{aligned}$$

The equation (7.12) is a consequence of the equation system (7.5). Hence $a_1(s)c_{j_1} + \cdots + a_{p-r}(s)c_{j_{p-r}} = c_{j_s}$. Let us denote $\varepsilon_{j_s} = a_1(s)\varepsilon_{j_1} + \cdots + a_{p-r}(s)\varepsilon_{j_{p-r}}$. Then

$$\begin{aligned} |\varepsilon_{j_s}| &\le |a_1(s)|\,|\varepsilon_{j_1}| + \cdots + |a_{p-r}(s)|\,|\varepsilon_{j_{p-r}}| \\ &< (|a_1(s)| + \cdots + |a_{p-r}(s)|)\frac{\Delta}{3mnM} < \frac{\Delta}{3mn}. \end{aligned}$$

Thus, rational numbers d_u which satisfy condition (7.11) are found for all variables which are represented (with nonzero coefficients) in the equation system (8): for free variables condition (11) follows from inequality (13). For other variables x_u we can give to d_u any arbitrary rational value which satisfies condition (11).

III. Now we construct the following matrix $E = \|e_{ij}\|$ of rational numbers. We define $e_{ij} = d_u$ and $\varepsilon_{ij} = \varepsilon_u$ if x_{ij} is the variable x_u of system (7.5). In the opposite case we define $e_{ij} = 1$, $\varepsilon_{ij} = 0$.

Lemma 2. *The* $(\mathscr{A}, \mathbb{R}, C)$*-problem is equivalent to the* $(\mathscr{A}, \mathbb{Q}, E)$*-problem.*

Proof. We must demonstrate that $CU_p < CU_q$ if an only if $EU_p < EU_q$ for any two matrices U_p and U_q listed in (7.1).

If $CU_p = CU_q$, $p<q$, then $L_p = L_q$ and the system (7.8) must contain either the equation $f_q - f_p = 0$ or two equations $f_p - f_{k_\upsilon} = 0$ and $f_q - f_{k_\upsilon} = 0$ for some υ. Lemma 1 demonstrates that the numbers (7.10) satisfy system (7.8). Hence we have either

$$EU_p = f_p(d_1,\dots,d_w) = f_q(d_1,\dots,d_w) = EU_q$$

or

$$\begin{aligned} EU_p &= f_p(d_1,\dots,d_w) = f_{k_\upsilon}(d_1,\dots,d_w), \\ EU_q &= f_q(d_1,\dots,d_w) = f_{k_\upsilon}(d_1,\dots,d_w). \end{aligned}$$

Therefore, $CU_p = CU_q$ always implies $EU_p = EU_q$.

Let now $CU_p < CU_q$, where $U_p = \|u_{ij}^p\|$ $(i=1,\dots,m;\ j=1,\dots,n)$. Then (see the Lemma 1 and the definition of the numbers e_{ij})

$$EU_p = \sum_{i=1}^{m} \sum_{j=1}^{n} e_{ij} u_{ij}^p = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} u_{ij}^p + \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_{ij} u_{ij}^p = CU_p + \delta_p$$

where (see (7.11))

$$|\delta_p| \le |\varepsilon_{11}|\, u_{11}^p + \cdots + |\varepsilon_{mn}|\, u_{mn}^p \le |\varepsilon_{11}| + \cdots + |\varepsilon_{mn}| < \tfrac{1}{3}\Delta$$

In the same way we can demonstrate that $EU_q = CU_q + \delta_q$ where $|\delta q| < \frac{1}{3}\Delta$. But the number Δ was chosen so that always $L_p \le L_q - \Delta$. Hence

$$\begin{aligned} EU_p &= CU_p + \delta_p = L_p + \delta_p \le L_q - \Delta + \delta_p = (L_q + \delta_q) - (\Delta - \delta_p + \delta_q) \\ &= (CU_q + \delta_q) - (\Delta - \delta_p + \delta_q) = EU_q - (\Delta - \delta_p + \delta_q). \end{aligned}$$

But $\Delta - \delta_p + \delta_q > 0$ and therefore $EU_p < EU_q$. So $CU_p < CU_q$ always implies $EU_p < EU_q$.

IV. For the matrix $E = \|e_{ij}\|$ let $e_{ij} = m_{ij}/n_{ij}$, $m_{ij} \in \mathbb{Z}$, $n_{ij} \in \mathbb{Z}$, where m_{ij} and n_{ij} are irreducible. If n is the least natural number for which $n/n_{ij} \in \mathbb{Z}$ for any n_{ij} then we define $F = \|f_{ij}\|$, $f_{ij} = ne_{ij}$. So all $f_{ij} \in Z$. It is clear that $FU = nEU$ for any $U \in \mathscr{A}$. Hence the $(\mathscr{A}, \mathbb{Q}, E)$- and $(\mathscr{A}, \mathbb{Z}, F)$-problems are equivalent. Therefore the use of Lemma 2 completes the proof of the theorem.

8. Equivalence theorem for real weight matrices with positive elements

Let $\mathbb{N}_M = \{1, \dots, M \mid 1 < \cdots < M\}$ be the finite ordered semigroup with the following addition $\oplus$: $a \oplus b = a + b$ if $a + b \le M$ and $a \oplus b = M$, if $a + b \ge M$. Let $\mathbb{R}^+$ be the additive semigroup of all positive real numbers, ordered in the usual way.

Theorem 5. *For any pair $m, n \in \mathbb{N}$ there exists a natural number M for which each $(\mathscr{A}, \mathbb{R}^+, C)$-problem is equivalent to an $(\mathscr{A}, \mathbb{N}_M, D)$-problem.*

Proof. In the trivial case that $CU_1 = \cdots = CU_k$, the $(\mathscr{A}, \mathbb{R}^+, C)$-problem is equivalent to each $(\mathscr{A}, \mathbb{N}_1, D)$-problem.

In the nontrivial case, let us consider a single $(\mathscr{A}, \mathbb{R}^+, C)$-problem first. For this problem we can repeat the proof of Theorem 4, choosing supplementarily all numbers ε_{ij} such that $|\varepsilon_{ij}|$ is smaller than $\min_{i,j} c_{ij}$. In such a way we prove that our $(\mathscr{A}, \mathbb{R}^+, C)$-problem is equivalent to an $(\mathscr{A}, \mathbb{N}, E)$-problem.

Let $T = \max_{1 \le l \le k} EU_l$. It is clear that $e_{ij} \le T$ if $u_{ij}^l \ne 0$ for some $l = 1, \dots, k$. But in the case that $u_{ij}^l = 0$ for all $l = 1, \dots, k$ we have chosen $e_{ij} = 1 \le T$. Hence all $e_{ij} \le T$ and therefore the $(\mathscr{A}, \mathbb{N}, E)$-problem is equivalent to the $(\mathscr{A}, \mathbb{N}_T, E)$-problem.

Let us now consider all $(\mathscr{A}, \mathbb{R}^+, C)$-problems with distance matrices of size $m \times n$. The number of such problems is infinite, but only a finite number of quasiorders on the set $\mathscr{A}$ correspond to them. Let $\sigma_1, \dots, \sigma_\kappa$ be the full list of different quasi-orders on the set $\mathscr{A}$ which correspond to $(\mathscr{A}, \mathbb{R}^+, C)$-problems of a given size. More accurately, let $C_1, \dots, C_\kappa$ be the different distance matrices which produce the filled quasiorders.

As we have proved above, for each $\mu = 1, \dots, \kappa$ there exists a natural number T_μ such that the $(\mathscr{A}, \mathbb{R}^+, C_\mu)$-problem is equivalent to an $(\mathscr{A}, \mathbb{N}_{T_\mu}, D_\mu)$-problem. Let us consider all numbers $T_1, \dots, T_\kappa$ and let $M_0 = \max_{1 \le \mu \le \kappa} T_\mu$, $M = M_o^2$. Let $M + M_\mu$ be the first member in the sequence M, $M+1$, $M+2, \dots$, which is divisible by T_μ: $M + M_\mu = T_\mu \cdot g_\mu$. Then the numbers

$$g_\mu, 2g_\mu, \dots, T_\mu g_\mu = M + M_\mu \tag{8.1}$$

form in $\mathbb{N}_M$ a subsemigroup order preserving isomorphic to the semigroup $\mathbb{N}_{T_\mu}$. Therefore if we transform each matrix D_μ over $\mathbb{N}_{T_\mu}$ into a matrix D'_μ by corresponding elements (8.1) of the semigroup $\mathbb{N}_M$. Then we obtain an $(\mathscr{A}, \mathbb{N}_M, D'_\mu)$-problem equivalent to the $(\mathscr{A}, \mathbb{N}_{T_\mu}, C_\mu)$-problem. Hence each $(\mathscr{A}, \mathbb{R}^+, C($-problem is equivalent to an $(\mathscr{A}, \mathbb{N}_M, D)$-problem, where $D = D'_\mu$ if the $(\mathscr{A}, \mathbb{R}^+, C)$-problem produces the quasiorder σ_μ on the set $\mathscr{A}$.

9. Bibliographical remarks

Weak equivalence was also studied in [2]. Given the terminology of [1] and [2], our notion of equivalence should be called strong equivalence. But we prefer our terminology and feel that our theorems demonstrate that equivalence in our sense is a working concept.

The equivalence problem:

Is each $\mathscr{A}$-problem equivalent to an $\mathscr{A}$-problem over a finite ordered semigroup? was stated in [3–5] for the case of a permutation set $\mathscr{A}$. Its positive solution for some special classes of ordered semigroups was given in [4] and for the general case in [6, 7]. A more strong version of the Theorem 4 was proved for a permutation set in [8] and the Theorem 5 in [7].

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